PARTIAL RESULTS ON COMMON ROOTS OF POSITIVE DERIVATIVES OF GEOMETRIC SUMS

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ABSTRACT. For positive integers d and r, let $H_{d,r}(z) = \sum_{j=0}^{d} \binom{r+j}{j} z^j$, a normalized positive derivative of a geometric sum. We investigate the conjecture that $H_{d,r}$ and $H_{e,s}$ are coprime. We establish this conjecture in several cases, including when $d < e$ and $r \geq s$. We thereby prove Scherbak's conjecture that truncated binomial polynomials $P_{N,k}$ and $P_{N,\ell}$ have no common roots when $0 < k < \ell < N$. We also study the analytic and arithmetic properties of the roots of $H_{d,r}$. Finally, we show that every fixed complex number is a root of $H_{d,r}$ for only finitely many pairs (d, r).

1. INTRODUCTION

Let $G_n(z) = 1 + z + \cdots + z^n$ be the geometric sum of degree n. For integers $d, r \geq 1$, put

$$H_{d,r}(z) = \sum_{j=0}^{d} \binom{r+j}{j} z^j. \tag{1.1}$$

Since

$$H_{d,r}(z) = \frac{1}{r!} G_{d+r}^{(r)}(z), \tag{1.2}$$

these polynomials are precisely the nonconstant positive derivatives of geometric sums, normalized to have constant coefficient 1. We investigate whether distinct members of this two-parameter family can share a complex root.

Conjecture 1.1 (No common root)**.** *For any two distinct pairs $(d, r), (e, s) \in \mathbb{Z}_{\geq 1}^2$,*

$$\gcd_{\mathbb{Q}[z]}(H_{d,r}, H_{e,s}) = 1.$$

Equivalently, $H_{d,r}$ and $H_{e,s}$ have no common root in $\mathbb{C}$.

We begin by reviewing the history and motivation for Conjecture 1.1. Borisov, Filaseta, Lam, and Trifonov [1, Conjectures 1–2] conjectured that $H_{d,r}$ is irreducible for $d \geq 1$ and $r \geq 1$. Obviously, if every such $H_{d,r}$ is irreducible, then Conjecture 1.1 follows.

In the literature, researchers often study an equivalent, better-known formulation in terms of truncated binomial expansions rather than studying the irreducibility of $H_{d,r}$ directly.

$$P_{N,k}(w) = \sum_{j=0}^{k} \binom{N}{j} w^j, \qquad 1 \leq k \leq N-2.$$

Indeed, Vandermonde's identity gives

$$(1+w)^d H_{d,r}\left(\frac{w}{1+w}\right) = P_{d+r+1,d}(w). \tag{1.3}$$

The substitution $z = w/(1+w)$ gives a bijection between the zero sets of these polynomials. To see this, let β be a root of $P_{d+r+1,d}$. Since $P_{N,k}(-1) = (-1)^k \binom{N-1}{k} \neq 0$, we have $\beta \neq -1$,

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so $\alpha = \frac{\beta}{1+\beta}$ is a root of $H_{d,r}$. Conversely, let α be a root of $H_{d,r}$. Since $H_{d,r}(1) > 0$, we have $\alpha \neq 1$, and $\beta = \frac{\alpha}{1-\alpha}$ is a root of $P_{N,k}$. Moreover, $\mathbb{Q}(\alpha) = \mathbb{Q}(\beta)$. Thus,

$$\begin{aligned} H_{d,r} \text{ is irreducible over } \mathbb{Q} &\iff [\mathbb{Q}(\alpha) : \mathbb{Q}] = \deg H_{d,r} = d \\ &\iff [\mathbb{Q}(\beta) : \mathbb{Q}] = \deg P_{d+r+1,d} = d \\ &\iff P_{d+r+1,d} \text{ is irreducible over } \mathbb{Q}. \end{aligned}$$

Motivated by Schubert calculus, Scherbak [8, Conjecture 3] proposed the following conjecture; see also [3].

Conjecture 1.2 (Scherbak)**.** *Truncated binomial polynomials $P_{N,k}$ and $P_{N,\ell}$ have no common roots when $0 < k < \ell < N$.*

[3] states that Scherbak proved the conjecture. However, the older arXiv version of [8] states the conjecture as a theorem, whereas the latest arXiv version states it as a conjecture. To the best of the author's knowledge, Conjecture 1.2 remains open.

Under the correspondence in (1.3), the case $k < \ell$ with fixed N becomes

$$(d, r) = (k, N - k - 1), \qquad (e, s) = (\ell, N - \ell - 1),$$

so $d < e$ and $r > s$. This case is therefore covered by part (ii) of Theorem 1.3 below and Remark 3.6. Thus, our results prove Scherbak's conjecture, and Conjecture 1.1 generalizes it.

There are several reasons to state Conjecture 1.1 in terms of $H_{d,r}$ rather than $P_{N,k}$. First, the parameter space for $H_{d,r}$ is more convenient to describe: it is given by $d \geq 1$ and $r \geq 1$. Under (1.3), the corresponding parameter space for $P_{N,k}$ is the triangular lattice $\{(N, k) \in \mathbb{Z}^2 : N \geq 3, \quad 1 \leq k \leq N - 2\}$. Second, the proofs are more transparent in terms of $H_{d,r}$ than in terms of $P_{N,k}$. The polynomial $H_{d,r}$ has a simple relation to $B_{d,r}$, defined in Lemma 2.1, and its roots are amenable to complex-analytic study. Let $F_{d,r}(X) = X^d H_{d,r}(1/X)$ be the reversed polynomial. The polynomial $F_{d,r}$ is monic, and its roots are algebraic integers. We can therefore study their arithmetic properties using p-adic valuations.

Before stating our main results, we recall previous work on the irreducibility of $P_{N,k}$. Filaseta, Kumchev, and Pasechnik [3, Theorem 3] proved that for fixed k, the polynomial $P_{N,k}$ is irreducible when n is sufficiently large. Khanduja, Khassa, and Laishram [5, Theorem 1.1] proved that $P_{N,d}$ is irreducible over $\mathbb{Q}$ whenever $2d \leq N < (d + 1)^3$. This result was subsequently strengthened by Laishram and Yadav [7, Theorem 1.1] to that $P_{N,d}$ is irreducible over $\mathbb{Q}$ when $2d \leq N < (d+1)^{10}$. For $1 \leq d \leq 6$, Dubickas and Šiurys [2, Theorem 1.1] proved that $H_{d,r}$ is irreducible over $\mathbb{Q}$ for every $r \geq 1$. Moreover, Filaseta and Moy [4, Theorem 1] showed that, for each fixed integer $d \geq 2$ with $d \neq 6$, the Galois group of $H_{d,r}$ over $\mathbb{Q}$ is isomorphic to S_d for all sufficiently large r. The remaining case $d = 6$ was subsequently established by Klahn and Technau [6, Theorem 2].

Main results. Our first theorem combines complex-analytic and arithmetic methods to establish coprimality in several parameter regions. The approach compares pairs of polynomials directly and does not assume that either polynomial is irreducible.

Theorem 1.3. *Let d, r, e, s be positive integers, with $(d, r) \neq (e, s)$. The polynomials $H_{d,r}$ and $H_{e,s}$ are coprime in each of the following cases:*

(i) $d = e$;
(ii) $d < e$ *and* $r \geq s$;
(iii) $d < e$, $r < s$, *and there is a prime p with* $\max(e, s) < p \leq e + s + 1$;
(iv) $\gcd(\binom{d+r}{d}, \binom{e+s}{e}) = 1$;
(v) $e = d + 1, s = r + 1, r + 2$.

Parts (i) and (ii) are proved in Section 3 using analytic properties of the roots. Part (i) follows from the Eneström–Kakeya theorem, while part (ii) uses complex-analytic techniques

often used to study starlike functions. In fact, the author first proved Scherbak's no-common-root conjecture 1.2 using the Eneström–Kakeya theorem and then proposed the general no-common-root conjecture 1.1. Although Scherbak's conjecture is a special case of part (ii), the Eneström–Kakeya theorem is no longer the main ingredient in the proof of part (ii).

Part (iii) is proved in Section 4 using arithmetic properties of the roots. Parts (iv) and (v) are proved by elementary arguments in Section 5.

Our second main result concerns the recurrence of individual roots throughout the family.

Theorem 1.4 (Finite root recurrence)**.** *For any nonzero polynomial $P \in \mathbb{C}[z]$, only finitely many pairs $(d, r) \in \mathbb{Z}^2_{\geq 1}$ satisfy*

$$\deg \gcd_{\mathbb{C}[z]}(P, H_{d,r}) > 0.$$

In particular, every fixed complex number is a zero of only finitely many members of the family.

Consequently, each fixed $H_{d,r}$ shares a root with only finitely many other members. This is weaker than Conjecture 1.1, but it holds without any assumption of irreducibility. Its proof in Section 6 uses the prime number theorem and a congruence for $F_{d,r}$ modulo suitably chosen primes.

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2. Complex property of roots and identities

For $d, r \in \mathbb{Z}_{\geq 1}$, define

$$H_{d,r}(z) = \sum_{j=0}^{d} \binom{r+j}{j} z^j.$$

Put

$$N = d + r + 1, \qquad C_{d,r} = \frac{N!}{d!r!}, \qquad F_{d,r}(X) = X^d H_{d,r}(1/X). \tag{2.1}$$

The reciprocal polynomial $F_{d,r}$ is monic and belongs to $\mathbb{Z}[X]$. Its endpoint values are

$$F_{d,r}(0) = \binom{N-1}{d}, \qquad F_{d,r}(1) = \binom{N}{d}. \tag{2.2}$$

Since all constant terms of $H_{d,r}(z)$ are nonzero, taking reciprocals preserves factor degrees and the property of having a common root.

Lemma 2.1 (Differential and integral identities)**.** *The polynomials above satisfy*

$$(1-z)H'_{d,r}(z) = (r+1)H_{d,r}(z) - C_{d,r}z^d, \tag{2.3}$$

$$X(X-1)F'_{d,r}(X) = (dX - N)F_{d,r}(X) + C_{d,r}. \tag{2.4}$$

Define

$$B_{d,r}(z) = 1 - (1-z)^{r+1}H_{d,r}(z).$$

Then

$$B_{d,r}(z) = C_{d,r}\int_0^z t^d(1-t)^r\,dt, \tag{2.5}$$

$$B_{d,r}(z) = 1 - B_{r,d}(1-z) = z^{d+1}H_{r,d}(1-z). \tag{2.6}$$

The integral in (2.5) *is a polynomial antiderivative and is therefore independent of the path. Every root of $H_{d,r}$ is simple and is different from* 0 *and* 1*.*

Proof. For $a_j = \binom{r+j}{j}$, the relation $(j+1)a_{j+1} = (r+j+1)a_j$ proves (2.3) by comparing coefficients, including the leading coefficient. Substitution of $z = 1/X$ gives (2.4). Differentiating $B_{d,r}$ gives $B'_{d,r}(z) = C_{d,r}z^d(1-z)^r$. Together with $B_{d,r}(0) = 0$, we have (2.5). The beta integral $\int_0^1 t^d(1-t)^r\,dt = d!r!/(d+r+1)!$ and the substitution $t \mapsto 1-t$ give (2.6). The

values $H_{d,r}(0)$ and $H_{d,r}(1)$ are positive. Hence, $z = 0, 1$ are not roots of $H_{d,r}(z)$. If a root α were multiple, (2.3) would give $C_{d,r}\alpha^d = 0$, a contradiction. Hence, every root of $H_{d,r}$ is simple. □

Remark 2.2. The differential identity (2.3) and the simplicity of the zeros of $H_{d,r}$ are already known in the froms of truncated-binomial polynomials $P_{N,k}(w)$ under the transformation (1.3). Indeed, Scherbak [8, Lemma 6] used

$$NP_{N,k}(w) - (1+w)P'_{N,k}(w) = N\binom{N-1}{k}w^k$$

to prove that truncated-binomial polynomials $P_{N,k}$ has only simple zeros for $k < N$; see also [3, Introduction]. Under the change of variables (1.3), with $N = d + r + 1$ and $k = d$, this identity is equivalent to (2.3).

The following root bounds were essentially established by Wagner [9, Proposition 6.3] for the H-polynomials of uniform matroids, which coincide with our polynomials $H_{d,r}$ after a change of parameters. We include a proof for completeness, together with the strictness of the bounds when $d \geq 2$.

Lemma 2.3 (Eneström–Kakeya coefficient-ratio bounds)**.** *If $A(z) = \sum_{j=0}^{m} a_j z^j$, with $m \geq 1$ and all $a_j > 0$, then every root satisfies*

$$\min_{j<m} \frac{a_j}{a_{j+1}} \leq |z| \leq \max_{j<m} \frac{a_j}{a_{j+1}}. \tag{2.7}$$

Consequently every root α of $H_{d,r}$ satisfies

$$\frac{1}{r+1} \leq |\alpha| \leq \frac{d}{d+r} < 1. \tag{2.8}$$

Both nontrivial inequalities are strict when $d \geq 2$; for $d = 1$ the only root is $-1/(r+1)$.

Proof. Let $R = \max_{0\leq j<m} a_j/a_{j+1}$ and $b_j = a_j R^j$. Then $b_j \geq b_{j-1}$ for $1 \leq j \leq m$, and

$$(1-w)A(Rw) = b_0 + \sum_{j=1}^{m}(b_j - b_{j-1})w^j - b_m w^{m+1}.$$

For $|w| > 1$,

$$\begin{aligned}
\left| b_0 + \sum_{j=1}^{m}(b_j - b_{j-1})w^j \right| &\leq b_0 + \sum_{j=1}^{m}(b_j - b_{j-1})|w|^j \\
&\leq b_0 + \sum_{j=1}^{m}(b_j - b_{j-1})|w|^m \\
&= b_0 + b_m|w|^m - b_0|w|^m \\
&\leq b_m|w|^m.
\end{aligned} \tag{2.9}$$

Since $|b_m w^{m+1}| = b_m|w|^{m+1} > b_m|w|^m$, we have $|(1-w)A(Rw)| \neq 0$ for $|w| > 1$. This proves the upper bound. For the lower bound, consider the reversed polynomial

$$A^*(z) := z^m A(1/z) = \sum_{j=0}^{m} a_{m-j} z^j.$$

If α is a root of A, then $1/\alpha$ is a root of A^*. Applying the upper bound to A^* gives the lower bound of A as required (2.7).

For $H_{d,r}$ the consecutive ratios are $(j+1)/(r+j+1)$, strictly increasing with j. Applying (2.7) to $H_{d,r}$ gives (2.8).

If $d \geq 2$, the root reaches the upper bound if and and only if $|w| = 1$. It forces all inequality in (2.9) are equality. By the following lemma 2.4, we have

$$0 = \arg(b_0) \equiv \arg((b_1 - b_0)w) \mod 2\pi,$$

which implies $\arg(w) = 0$, hence, $w = 1$. However, the polynomial is positive there, contradiction. For the equality in the lower bound, similarly consider the reversed polynomial. □

Lemma 2.4. *Let $z_1, z_2, \ldots, z_n$ be nonzero complex numbers. If*

$$|z_1 + z_2 + \cdots + z_n| = |z_1| + |z_2| + \cdots + |z_n|,$$

then

$$\arg(z_1) \equiv \arg(z_2) \equiv \cdots \equiv \arg(z_n) \pmod{2\pi}.$$

Proof. This follows from a simple algebraic calculation. Alternatively, one may argue geometrically. Identify each z_i with a nonzero vector $v_i \in \mathbb{R}^2 \cong \mathbb{C}$. Then

$$|v_1 + v_2 + \cdots + v_n| = |v_1| + |v_2| + \cdots + |v_n|$$

implies that all the vectors v_i point in the same direction. □

Proposition 2.5 (The least root modulus)**.** *Let $\tau_{d,r} > 0$ be determined by*

$$J_{d,r}(\tau_{d,r}) = 1, \qquad J_{d,r}(t) = C_{d,r} \int_0^t y^d (1+y)^r \, dy. \tag{2.10}$$

Every root α of $H_{d,r}$ satisfies $|\alpha| \geq \tau_{d,r}$. If d is odd, $-\tau_{d,r}$ is the unique real root and the unique root of minimum modulus. If d is even, there are no real roots and every root satisfies $|\alpha| > \tau_{d,r}$.

Proof. The function $J_{d,r}$ is continuous and strictly increases from 0 to ∞, so (2.10) uniquely determines the value $\tau_{d,r}$. At a root α, by chaning variable $t = \alpha u$, (2.5) gives

$$\begin{aligned} J_{d,r}(\tau_{d,r}) = 1 = B_{d,r}(\alpha) &\leq C_{d,r}|\alpha|^{d+1} \int_0^1 u^d |1 - \alpha u|^r \, du \\ &\leq C_{d,r}|\alpha|^{d+1} \int_0^1 u^d (1 + |\alpha| u)^r \, du = J_{d,r}(|\alpha|). \end{aligned}$$

Since $J_{d,r}$ is continuous and strictly increases from 0 to ∞, we have $|\alpha| \geq \tau_{d,r}$. If $|\alpha| = \tau_{d,r}$, the second inequality is an equality, which implies $|1 - \alpha u| = 1 + |\alpha| u$ for $0 < u \leq 1$. By lemma 2.4, we have $0 = \arg(1) \equiv \arg(-\alpha u) \mod 2\pi$. Hence, α is a negative real number.

For $t \geq 0$, making the change of variables $y = -u$, we obtain

$$\begin{aligned} B_{d,r}(-t) &= C_{d,r} \int_0^{-t} u^d (1-u)^r \, du \\ &= C_{d,r}(-1)^{d+1} \int_0^t y^d (1+y)^r \, dy \\ &= (-1)^{d+1} J_{d,r}(t). \end{aligned}$$

Hence,

$$(1+t)^{r+1} H_{d,r}(-t) = 1 - (-1)^{d+1} J_{d,r}(t). \tag{2.11}$$

If d is odd, we have $(1+t)^{r+1} H_{d,r}(-t) = 1 - J_{d,r}(t)$. Hence, $-\tau_{d,r}$ is the unique real root of $H_{d,r}(z)$ and which is also the only zero attaining the lower bound. If d is even, we have $(1+t)^{r+1} H_{d,r}(-t) = 1 + J_{d,r}(t) > 0$. Hence, $H_{d,r}(z)$ has no negative real roots. $H_{d,r}(z)$ has no potive real roots since coefficients of $H_{d,r}$ is positive. Hence, $H_{d,r}(z)$ has no real roots and the lower bound is strict. □

3. Coprimality by parameter ordering

3.1. **Equal degrees.**

Theorem 3.1. *For $d \geq 1$ and distinct positive integers r, s, $\gcd(H_{d,r}, H_{d,s}) = 1$.*

Proof. We may assume $v := s - r \geq 1$. Pascal's identity gives

$$(1-z)H_{d,t+1}(z) = H_{d,t}(z) - \binom{d+t+1}{d} z^{d+1}.$$

Multiplying by successive powers of $1 - z$ and summing yields

$$(1-z)^v H_{d,s}(z) = H_{d,r}(z) - z^{d+1}T(1-z), \quad T(w) = \sum_{j=0}^{v-1} \binom{d+r+j+1}{d} w^j. \tag{3.1}$$

A common root α is nonzero and would satisfy $T(1-\alpha) = 0$. If $v = 1$, this is impossible because T is a positive constant. If $v \geq 2$, its consecutive coefficient ratios are

$$\frac{r+j+2}{d+r+j+2} \qquad (0 \leq j \leq v-2).$$

Lemma 2.3, applied to T and to $H_{d,s}$, gives

$$|1-\alpha| \leq \frac{s}{d+s}, \qquad |\alpha| \leq \frac{d}{d+s}.$$

The two upper bounds sum to one. Equality throughout $1 \leq |\alpha| + |1-\alpha| \leq 1$ forces α to lie in $[0,1]$, where $H_{d,s}$ is positive since coefficients of $H_{d,s}$ is positive. This is a contradiction. □

3.2. **An analytic comparison lemma.**

Lemma 3.2. *Let f, g be holomorphic on a neighborhood of $|z| \leq R$, where $R > 0$. Suppose that g vanishes at $z = 0$ to order $m \geq 1$ and has no other zero on that closed disk, that g' has no zero on $0 < |z| \leq R$, and that f vanishes at $z = 0$ to order greater than m. If*

$$\operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0, \qquad \left|\frac{f'(z)}{g'(z)}\right| < 1 \quad (0 < |z| \leq R), \tag{3.2}$$

then $|f(z)| < |g(z)|$ for $0 < |z| \leq R$.

Proof. One can check that the function $A = zg'/g$ extends holomorphically to zero with $A(0) = m$, is nonzero on the closed disk, and satisfies

$$A + \frac{zA'}{A} = 1 + \frac{zg''}{g'}. \tag{3.3}$$

We first show $\operatorname{Re} A > 0$ on the neighborhood of $|z| \leq R$. Otherwise, let ρ_0 be the minimal radius at which $\operatorname{Re} A$ reaches 0. There exists a boundary point $|z_0| = \rho_0$ such that $\operatorname{Re} A(z_0) = 0$. By definition,

$$\operatorname{Re} A(z) \geq 0 \quad \text{for } |z| \leq |z_0| = \rho_0.$$

On the boundary $|z| = \rho_0$, write $z = \rho e^{i\theta}$, $z_0 = \rho_0 e^{i\theta_0}$. The function $\operatorname{Re} A(\rho_0 e^{i\theta})$ attains its minimum at $\theta = \theta_0$ by the definition of ρ_0. By the chain rule, we have

$$\frac{d}{d\theta} \operatorname{Re}(A(\rho_0 e^{i\theta})) = \operatorname{Re}\left(A'(\rho_0 e^{i\theta}) \cdot i\rho_0 e^{i\theta}\right).$$

Therefore, $\operatorname{Re}(iz_0 A'(z_0)) = 0$, which means $\operatorname{Im}(z_0 A'(z_0)) = 0$. Hence, $z_0 A'(z_0)$ is a real number. Both $A(z_0)$ and $\frac{z_0 A'(z_0)}{A(z_0)}$ are purely imaginary, which contradicts (3.2) in view of (3.3).

Put $q = f/g$, extending it holomorphically to $z = 0$. Since f vanishes there to order greater than m, while g vanishes to order m, we have $q(0) = 0$. Assume that $|q(z)|$ can reach 1. Let

ρ_1 be the minimal radius at which $|q(z)|$ reaches 1. There exists a boundary point z_1, with $|z_1| = \rho_1$, such that $|q(z_1)| = 1$. By definition,

$$|q(z)| \le 1 \qquad \text{for } |z| \le |z_1| = \rho_1.$$

Write $z = \rho e^{i\theta}$, $z_1 = \rho_1 e^{i\theta_1}$. When we fix $\rho = \rho_1$, the function $|q(\rho_1 e^{i\theta})|$ attains its maximum at $\theta = \theta_1$. Hence,

$$\frac{d}{d\theta}|q(\rho_1 e^{i\theta})|\Big|_{\theta=\theta_1} = 0.$$

When we fix $\theta = \theta_1$, by the definition of ρ_1, we have $|q(\rho e^{i\theta_1})| < |q(\rho_1 e^{i\theta_1})|$ for $\rho < \rho_1$. Hence,

$$\frac{d}{d\rho}|q(\rho e^{i\theta_1})|\Big|_{\rho=\rho_1} = \lim_{\rho\to\rho_1^-} \frac{|q(\rho_1 e^{i\theta_1})| - |q(\rho e^{i\theta_1})|}{\rho_1 - \rho} \ge 0.$$

By the following lemma 3.4,

$$0 = \frac{d}{d\theta}|q(\rho_1 e^{i\theta})|\Big|_{\theta=\theta_1} = \operatorname{Re}\left(\frac{q'(z_1)\, iz_1}{q(z_1)}\right)|q(z_1)| = -\operatorname{Im}\left(\frac{q'(z_1)z_1}{q(z_1)}\right),$$

$$0 \le \frac{d}{d\rho}|q(\rho e^{i\theta_1})|\Big|_{\rho=\rho_1} = \operatorname{Re}\left(\frac{q'(z_1)e^{i\theta_1}}{q(z_1)}\right)|q(z_1)| = \operatorname{Re}\left(\frac{q'(z_1)e^{i\theta_1}}{q(z_1)}\right).$$

Hence, $\operatorname{Re}\left(\frac{q'(z_1)z_1}{q(z_1)}\right) = \rho_1 \operatorname{Re}\left(\frac{q'(z_1)e^{i\theta_1}}{q(z_1)}\right) \ge 0$. Consequently, $\lambda := \frac{q'(z_1)z_1}{q(z_1)}$ is real and nonnegative. In other words, $\lambda \ge 0$. Since $f = qg$ and $|q(z_1)| = 1$, one may check that

$$\begin{aligned}\left|\frac{f'(z_1)}{g'(z_1)}\right|^2 &= \left|1 + \frac{\lambda}{A(z_1)}\right|^2 \\ &= 1 + \frac{\lambda}{A(z_1)}\frac{\lambda}{\overline{A(z_1)}} + \lambda\left(\frac{1}{A(z_1)} + \frac{1}{\overline{A(z_1)}}\right) \\ &= 1 + \frac{\lambda^2}{|A(z_1)|^2} + 2\lambda \operatorname{Re}\left(\frac{1}{A(z_1)}\right) \\ &\ge 1,\end{aligned}$$

again a contradiction to (3.2). □

Remark 3.3. The first part of the proof shows that $\operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0$ implies that $\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) > 0$. The remainder of the argument uses only the latter condition. Thus, the key idea derives from the theory of starlike functions. In particular, the real-part condition in Lemma 3.2 can be weakened to $\operatorname{Re}\left(\frac{zg'(z)}{g(z)}\right) > 0$. However, for the applications considered here, it is easier to verify the stronger condition $\operatorname{Re}\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0$, as in Theorem 3.5.

The following lemma is a easy fact from complex analysis.

Lemma 3.4. *Let $q(z)$ be a holomorphic function, and let $z = \rho e^{i\theta}$ with $\rho > 0$. Assume $q(z) \ne 0$. Then*

$$\frac{d}{d\rho}|q(\rho e^{i\theta})| = \operatorname{Re}\left(\frac{q'(\rho e^{i\theta})e^{i\theta}}{q(\rho e^{i\theta})}\right)|q(\rho e^{i\theta})|,$$

$$\frac{d}{d\theta}|q(\rho e^{i\theta})| = \operatorname{Re}\left(\frac{q'(\rho e^{i\theta})i\rho e^{i\theta}}{q(\rho e^{i\theta})}\right)|q(\rho e^{i\theta})|.$$

Proof. Let $z = z(t)$ for a real parameter t. Then

$$\begin{aligned}\frac{d}{dt}|q(z(t))|^2 &= \frac{d}{dt}(q(z(t))\overline{q(z(t))}) \\ &= \frac{dq(z(t))}{dt}\,\overline{q(z(t))} + q(z(t))\frac{d}{dt}\overline{q(z(t))} \\ &= q'(z(t))z'(t)\overline{q(z(t))} + q(z(t))\overline{q'(z(t))}\,\overline{z'(t)} \\ &= 2\,\mathrm{Re}\Big(q'(z(t))z'(t)\overline{q(z(t))}\Big).\end{aligned}$$

On the other hand,

$$\frac{d}{dt}|q(z(t))|^2 = 2|q(z(t))|\frac{d}{dt}|q(z(t))|.$$

Hence,

$$\frac{d}{dt}|q(z(t))| = \frac{\mathrm{Re}\Big(q'(z(t))z'(t)\overline{q(z(t))}\Big)}{|q(z(t))|}.$$

Using

$$\frac{\overline{q(z)}}{|q(z)|} = \frac{|q(z)|}{q(z)},$$

we obtain

$$\frac{d}{dt}|q(z(t))| = \mathrm{Re}\left(\frac{q'(z(t))z'(t)}{q(z(t))}\right)|q(z(t))|.$$

Therefore, substituting $z = \rho e^{i\theta}$ gives

$$\frac{d}{d\rho}|q(\rho e^{i\theta})| = \mathrm{Re}\left(\frac{q'(\rho e^{i\theta})e^{i\theta}}{q(\rho e^{i\theta})}\right)|q(\rho e^{i\theta})|,$$

$$\frac{d}{d\theta}|q(\rho e^{i\theta})| = \mathrm{Re}\left(\frac{q'(\rho e^{i\theta})i\rho e^{i\theta}}{q(\rho e^{i\theta})}\right)|q(\rho e^{i\theta})|.$$

□

Theorem 3.5. *If $d < e$ and $r \geq s \geq 1$, then* $\gcd(H_{d,r}, H_{e,s}) = 1$.

Proof. First assume $r > s$ and put $\rho = d/(d+r)$. The polynomial $B_{d,r}$ has no zero on $0 < |z| \leq \rho$. Indeed, (2.6) and Lemma 2.3 would give $|1-z| \leq r/(d+r) = 1-\rho$ at such a zero. Equality in $1 \leq |z| + |1-z| \leq 1$ then forces $z = \rho$, whereas $H_{r,d}(1-\rho) > 0$.

Apply Lemma 3.2 with $g = B_{d,r}$, $f = B_{e,s}$, and $R = \rho$. The orders at zero are $d+1$ and $e+1$. The derivative $g' = C_{d,r}z^d(1-z)^r$ has no nonzero zero in this disk. One may check that

$$\mathrm{Re}\left(1 + \frac{zg''}{g'}\right) = \mathrm{Re}\left(d+1 - \frac{rz}{1-z}\right) \geq d+1 - \frac{r\rho}{1-\rho} = 1.$$

Furthermore,

$$\left|\frac{f'}{g'}\right| \leq \frac{C_{e,s}}{C_{d,r}}\frac{\rho^{e-d}}{(1-\rho)^{r-s}} =: K(e,s). \tag{3.4}$$

The starting value is $K(d+1, r-1) = d/(d+1) < 1$. Increasing e or decreasing s within $e \geq d+1$, $1 \leq s \leq r-1$ decreases K, because

$$\frac{K(e+1,s)}{K(e,s)} = \rho\frac{e+s+2}{e+1} = \frac{d}{d+r}\frac{e+s+2}{e+1} = \frac{de+ds+2d}{de+d+re+r} = \frac{(de+d)+d(s+1)}{(de+d)+(e+1)r} < 1,$$

$$\frac{K(e,s-1)}{K(e,s)} = \frac{s}{(e+s+1)(1-\rho)} = \frac{s}{e+s+1}\frac{d+r}{r} = \frac{sd+sr}{er+sr+r} = \frac{sr+sd}{sr+r(e+1)} < 1.$$

Thus the lemma 3.2 gives $|B_{e,s}(z)| < |B_{d,r}(z)|$ on the punctured closed disk $0 < |z| \leq \rho$. By (2.8), the root of $H_{d,r}$ must lie in $0 < |z| \leq \rho$. A common root α of $H_{d,r}$ and $H_{e,s}$ make both values of $B_{d,r}(\alpha)$ and $B_{e,s}(\alpha)$ equal to one, a contradiction.

For $r = s$, subtract the truncations:

$$H_{e,r}(z) - H_{d,r}(z) = z^{d+1} \sum_{j=0}^{e-d-1} \binom{r+d+1+j}{d+1+j} z^j.$$

Let $A(z) = \sum_{j=0}^{e-d-1} \binom{r+d+1+j}{d+1+j} z^j$. If $e = d+1$, $A(z)$ is a nonzero constant. Since $z = 0$ is not a root of $H_{d,r}$, no common root exits. If $e > d+1$, by Lemma 2.3, all roots of $A(z)$ have modulus at least $(d+2)/(r+d+2)$. This exceeds $d/(d+r)$, the upper bound for the roots of $H_{d,r}$. Hence no common root is possible in this case either. □

Theorems 3.1 and 3.5 prove parts (i) and (ii) of Theorem 1.3. The exact remaining region, before applying additional sufficient criteria, is

$$d < e, \qquad r < s. \tag{3.5}$$

Remark 3.6. Scherbak's Conjecture 1.2 also containing the boundary case $\ell = N - 1$. Under the correspondence in (1.3), $P_{N,N-1}$ corresponds to $H_{N-1,0}(z) = 1 + z + \cdots + z^{N-1}$, while $P_{N,k}$ corresponds to $H_{k,N-k-1}$ for $1 \leq k \leq N-2$. Every zero of $H_{N-1,0}$ has modulus exactly 1, whereas, by Lemma 2.3, every zero of $H_{k,N-k-1}$ has modulus at most $k/(N-1) < 1$. The zero-set correspondence therefore shows that $P_{N,N-1}$ and $P_{N,k}$ have no common root. Together with Theorem 3.5, this proves Conjecture 1.2, including the boundary case $\ell = N-1$.

4. Arithmetic property of roots and one-prime criteria

For $d, r \geq 1$, recall the notation

$$N = d + r + 1, \qquad C_{d,r} = \frac{N!}{d!r!}, \qquad F_{d,r}(X) = X^d H_{d,r}(1/X).$$

In the previous section, we use complex analysis to study the analytic property of roots of $H_{d,r}$. However, to study the arithmetic property of roots of $H_{d,r}$, it is better to use the reversed polynomial $F_{d,r}$. The reason is the following. We normalize the p-adic valuation by $v_p(p) = 1$ and extend it to $\overline{\mathbb{Q}}_p$. Every root of a monic polynomial in $\mathbb{Z}[X]$ has nonnegative p-adic valuation. The reversed polynomial $F_{d,r}$ is monic polynomial and hence, its root are in ring of integer of a field extension of $\mathbb{Q}_p$. Hence, we can consider reduction of roots of $F_{d,r}$ modulo p.

Lemma 4.1 (Root valuations in two residue classes)**.** *Let $d, r \geq 1$, and let $p > d$ be a prime dividing $C_{d,r}$. Then there is a unique $k \in \{1, \ldots, d+1\}$ such that $p \mid r + k$, and set $h = v_p(r+k)$. Then*

$$F_{d,r}(X) \equiv X^{d-k+1}(X-1)^{k-1} \pmod{p}. \tag{4.1}$$

Hence, there are $(d-k+1)$-many roots of $F_{d,r}(X)$ equal to 0 after modulo p and there are $(k-1)$-many roots of $F_{d,r}(X)$ equal to 1 after modulo p.

More explicitly, every root $\beta \in \overline{\mathbb{Q}}_p$ of $F_{d,r}$ satisfies exactly one of

$$\bar{\beta} = 0, \quad v_p(\beta) = \frac{h}{d-k+1}; \tag{4.2}$$

$$\bar{\beta} = 1, \quad v_p(\beta - 1) = \frac{h}{k-1}, \tag{4.3}$$

where $\bar{\beta}$ is the reduction of β to the residue field.

The idea of the proof is essentially an application of the Newton polygon argument. The underlying arguments go back at least to [1, Section 2]; see also [3, Section 3] and [4, Section 4]. We formulate them in terms of two residue classes, which will be useful for comparing roots of distinct members of the family.

Proof. We have

$$C_{d,r} = \frac{(r+1)(r+2)\cdots(r+d+1)}{d!}.$$

The numerator interval has length $d+1 \le p$, and the denominator is a p-adic unit. This proves the existence and uniqueness of k.

Since $r \equiv -k \pmod{p}$ and $j!$ is invertible for $j \le d$, coefficient reduction gives

$$\binom{r+j}{j} \equiv (-1)^j \binom{k-1}{j} \pmod{p}.$$

Hence,

$$\begin{aligned} F_{d,r}(X) &= X^d \sum_{j=0}^{d} \binom{r+j}{j} \left(\frac{1}{X}\right)^j \\ &\equiv \sum_{j=0}^{d} (-1)^j \binom{k-1}{j} X^{d-j} \\ &= X^{d-k+1} \sum_{j=0}^{k-1} (-1)^j \binom{k-1}{j} X^{k-1-j} \\ &= X^{d-k+1}(X-1)^{k-1} \pmod{p}. \end{aligned}$$

The equality in third line follows from $\binom{k-1}{j} = 0$ for $j > k-1$.

Write

$$F_{d,r}(X) = \sum_{i=0}^{d} \binom{r+d-i}{d-i} X^i.$$

The binomial coefficients satisfy

$$v_p\left(\binom{r+d-i}{d-i}\right) = \begin{cases} h, & 0 \le i < d-k+1, \\ 0, & d \ge i \ge d-k+1. \end{cases}$$

If $v_p(\beta) > 0$, then

$$v_p\left(\binom{r+d-i}{d-i}\beta^i\right) > v_p\left(\binom{r+d}{d}\right) = h$$

for $0 < i < d-k+1$, and

$$v_p\left(\binom{r+d-i}{d-i}\beta^i\right) > v_p\left(\binom{r+k-1}{k-1}\beta^{d-k+1}\right) = (d-k+1)v_p(\beta),$$

for $d \ge i > d-k+1$. Since $F_{d,r}(\beta) = 0$, the minimum valuation among its terms must occur at least twice. Thus, $(d-k+1)v_p(\beta) = h$.

One have the following by using Vandermonde's identity,

$$F_{d,r}(1+Y) = \sum_{i=0}^{d} \binom{d+r+1}{d-i} Y^i. \tag{4.4}$$

One can check that

$$v_p\left(\binom{d+r+1}{d-i}\right) = \begin{cases} h, & 0 \le i < k-1, \\ 0, & d \ge i \ge k-1. \end{cases}$$

If $v_p(\beta-1) > 0$, the same minimum valuation argument gives $(k-1)v_p(\beta-1) = h$, proving (4.3). Finally (4.1) forces every root to reduce to one of these two distinct residues. □

Theorem 4.2 (One-prime separation)**.** *If $d < e$, $r < s$, and a prime satisfies*

$$\max(e,s) < p \le e+s+1, \tag{4.5}$$

then $\gcd(H_{d,r}, H_{e,s}) = 1$.

Proof. Put $n = d + r + 1$ and $m = e + s + 1$, so that $n < m$. Suppose that $F_{d,r}$ and $F_{e,s}$ have a common root β in $\overline{\mathbb{Q}}_p$.

If $p > n$, then

$$F_{d,r}(0) = \binom{n-1}{d} \not\equiv 0 \pmod{p}, \qquad F_{d,r}(1) = \binom{n}{d} \not\equiv 0 \pmod{p}.$$

On the other hand, Lemma 4.1 implies $F_{e,s}(X) \equiv X^{e-k+1}(X-1)^{k-1} \pmod{p}$ for where integer $k = p - s$. Hence, $F_{d,r} \pmod{p}$ and $F_{e,s} \pmod{p}$ are coprime to each other. Therefore, Thus no common root of $F_{d,r}$ and $F_{e,s}$ exists.

Now suppose $p \le n$. Since $p > \max(e, s)$, Lemma 4.1 applies to both polynomials, with $h = 1$ in each case.

If $\bar{\beta} = 0$, Lemma 4.1 gives

$$\frac{1}{n-p} = v_p(\beta) = \frac{1}{m-p},$$

contradicting $n < m$.

If $\bar{\beta} = 1$, Lemma 4.1 gives

$$\frac{1}{p-r-1} = v_p(\beta - 1) = \frac{1}{p-s-1},$$

contradicting $r < s$.

Since these are the only possible residue classes, the proof is complete. □

This completes the proof of part (iii) of Theorem 1.3. The proof shows that if the prime p satisfies $n < p \le m$, then coprimality follows directly by reducing both polynomials modulo p, without comparing the p-adic valuations of their roots. For the case $p \le n$, we have to use the arithmetic property of the roots.

Example 4.3. For $(d, r) = (18, 5)$ and $(e, s) = (20, 6)$, one can use $p = 23$. There is no prime p with $n < p \le m$, where $n = 24$ and $m = 27$.

5. Elementary coprimality criteria

Proposition 5.1 (Coprime leading coefficients)**.** *For positive integer parameters,b*

$$\gcd\left(\binom{d+r}{d}, \binom{e+s}{e}\right) = 1 \quad \Longrightarrow \quad \gcd(H_{d,r}, H_{e,s}) = 1.$$

Proof. Suppose that $H_{d,r}$ and $H_{e,s}$ have a common nonconstant irreducible factor. We may denote it as

$$P(X) = a_m X^m + a_{m-1} X^{m-1} + \cdots + a_0 \in \mathbb{Z}[X],$$

and degree $m \ge 1$.

The leading coefficients of $H_{d,r}$ and $H_{e,s}$ are $\binom{d+r}{d}$ and $\binom{e+s}{e}$, respectively, while both constant coefficients of $H_{d,r}$ and $H_{e,s}$ are 1. Consequently,

$$a_m \mid \binom{d+r}{d}, \qquad a_m \mid \binom{e+s}{e}, \qquad a_0 \mid 1.$$

Therefore, $a_m = \pm 1$, $a_0 = \pm 1$. Let $\alpha_1, \ldots, \alpha_m$ be the complex roots of P, counted with multiplicity. Since every root of P is also a root of $H_{d,r}$, inequation (2.8) gives root bounds

$$|\alpha_i| \le \frac{d}{d+r} < 1, \qquad 1 \le i \le m.$$

On the other hand, Vieta's formula yields $1 = |a_0| = |a_m| \prod_{i=1}^{m} |\alpha_i| < |a_m| = 1$, a contradiction. □

Proposition 5.2 (Small increasing shifts)**.** *For all $d, r \ge 1$,*

$$\gcd(H_{d,r}, H_{d+1,r+1}) = \gcd(H_{d,r}, H_{d+1,r+2}) = 1.$$

Proof. For the first shift, Pascal's identity gives

$$(1-z)H_{d+1,r+1}(z) - H_{d,r}(z) = z^{d+1}\left[\binom{d+r+1}{d+1} - \binom{d+r+2}{d+1}z\right].$$

A common root would therefore be the positive real number $(r+1)/(d+r+2)$, contrary to the positivity of the coefficients of $H_{d,r}$.

For the second shift, direct coefficient comparison gives

$$(1-z)^2 H_{d+1,r+2}(z) - H_{d,r}(z) = z^{d+1}R(z),$$

where

$$R(z) = \binom{d+r+3}{r+2}(z-1)^2 + \binom{d+r+2}{r+2}(z-1) - \binom{d+r+1}{r+1}.$$

Here $R(0) = \binom{d+r+1}{r} > 0$, $R(1) = -\binom{d+r+1}{r+1} < 0$, and $R(z) \to +\infty$ as $z \to +\infty$. Since R is quadratic, its two roots are positive: one lies in $(0,1)$ and the other in $(1,\infty)$. Neither can be a root of $H_{d,r}$.

□

6. Finite recurrence of roots

Lemma 6.1. *Let α be a complex number with $0 < |\alpha| < 1$, and define $U := \frac{1}{|\alpha|} - 1$. If α is a root for some $H_{d,r}$, then $\frac{r}{d} \le U$. Futhermore, there exist $\varepsilon > 0$ and d_0, depending on α, such that if $H_{d,r}(\alpha) = 0$ for $d \ge d_0$, then*

$$\varepsilon \le \frac{r}{d} \le U. \tag{6.1}$$

Proof. By equation (2.8), we have $|\alpha| \le \frac{d}{d+r} = \frac{1}{1+r/d}$, which gives the upper bound.

Putting $z = \alpha$ in equation (2.6), we obtain

$$\begin{aligned}
1 = |B_{d,r}(\alpha)| &= |\alpha|^{d+1}|H_{r,d}(1-\alpha)| \\
&= |\alpha|^{d+1}\Big|\sum_{j=0}^{r}\binom{d+j}{j}(1-\alpha)^j\Big| \\
&\le |\alpha|^{d+1}\sum_{j=0}^{r}\binom{d+j}{j}(1+|\alpha|)^j \\
&\le |\alpha|^{d+1}(1+|\alpha|)^r\sum_{j=0}^{r}\binom{d+j}{j} \\
&= |\alpha|^{d+1}(1+|\alpha|)^r\binom{d+r+1}{d+1}.
\end{aligned}$$

Choose a real number $t > 0$ such that $|\alpha|(1+t) < 1$. By the binomial theorem,

$$(1+t)^{d+r+1} = \sum_{j=0}^{d+r+1}\binom{d+r+1}{j}t^j \ge \binom{d+r+1}{r}t^r = \binom{d+r+1}{d+1}t^r.$$

Hence

$$\binom{d+r+1}{d+1} \le \frac{(1+t)^{d+r+1}}{t^r},$$

and therefore

$$\begin{aligned} 1 &\le |\alpha|^{d+1}(1+|\alpha|)^r \frac{(1+t)^{d+r+1}}{t^r} \\ &= \big(|\alpha|(1+t)\big)^{d+1} \left(\frac{(1+|\alpha|)(1+t)}{t}\right)^r \\ &= a^{d+1}b^r, \end{aligned}$$

where

$$a := |\alpha|(1+t) < 1, \qquad b := \frac{(1+|\alpha|)(1+t)}{t} > 1.$$

Choose $\varepsilon > 0$ small enough that $ab^\varepsilon < 1$. If $r/d < \varepsilon$, then

$$1 \le a^{d+1}b^r < a^{d+1}b^{\varepsilon d} = a(ab^\varepsilon)^d \longrightarrow 0 \qquad (d \to \infty),$$

a contradiction for all sufficiently large d. Thus $r/d \ge \varepsilon$ whenever d is sufficient large, which proves the lower bound. □

Theorem 6.2 (Finite root recurrence)**.** *For any nonzero polynomial $P \in \mathbb{C}[z]$, only finitely many pairs $(d, r) \in \mathbb{Z}^2_{\ge 1}$ satisfy*

$$\deg \gcd_{\mathbb{C}[z]}(P, H_{d,r}) > 0.$$

In particular, every fixed complex number is a zero of only finitely many members of the family.

Proof. Since a nonzero polynomial has only finitely many roots, it is enough to prove the assertion for a fixed $\alpha \in \mathbb{C}$. If α is not a root of any $H_{d,r}$, there is nothing to prove. Otherwise $\alpha \neq 0$ and $0 < |\alpha| < 1$.

Suppose that there are infinitely many pairs (d_i, r_i) such that $H_{d_i,r_i}(\alpha) = 0$. The upper bound in Lemma 6.1 implies that d_i is unbounded, for otherwise r would be bounded as well. We may assume $d_i \to \infty$ as $i \to \infty$.

By Lemma 6.1, there are constants $0 < \varepsilon \le U$ such that, for all sufficiently large d_i,

$$\varepsilon \le \frac{r_i}{d_i} \le U.$$

Set $M_i := \max(d_i, r_i)$, $\delta := \frac{1}{2}\min\{\varepsilon, U^{-1}\} > 0$. We have

$$\frac{\min(d_i, r_i)}{\max(d_i, r_i)} = \min\left\{\frac{r_i}{d_i}, \frac{d_i}{r_i}\right\} \ge \min\{\varepsilon, U^{-1}\} = 2\delta.$$

Hence

$$\begin{aligned} d_i + r_i + 1 &= \max(d_i, r_i) + \min(d_i, r_i) + 1 \\ &\ge M_i + 2\delta M_i + 1 \\ &\ge (1 + 2\delta)M_i. \end{aligned}$$

By the prime number theorem, for all sufficiently large M_i, there exists a prime p_i such that $M_i < p_i \le (1+\delta)M_i$. In particular, $\max(d_i, r_i) < p_i \le d_i + r_i + 1$. By Lemma 4.1, we have

$$F_{d_i,r_i}(X) \equiv X^{d_i+r_i+1-p_i}(X-1)^{p_i-r_i-1} \pmod{p_i}.$$

Set $\beta = 1/\alpha$. Let $A \in \mathbb{Z}[X]$ be the monic minimal polynomial of β. Then A is a nonconstant monic integral divisor of F_{d_i,r_i}. Consequently, $A(0) \equiv 0 \pmod{p_i}$ or $A(1) \equiv 0 \pmod{p_i}$, so

$$p_i \mid A(0)A(1).$$

Since A is the minimal polynomial of β and $|\beta| > 1$, neither X nor $X - 1$ divides A. Hence $A(0)A(1)$ is a fixed nonzero integer. However, as $d_i \to \infty$ and $p_i > d_i$, there are infinitely many distinct such primes p_i. This contradicts the fact that $A(0)A(1)$ is a fixed nonzero integer.

□

Declaration of AI Use

AI tools were used to identify relevant references, simplify arguments, and assist with manuscript preparation. In particular, the proofs of parts (i), (iv), and (v) of Theorem 1.3 were developed with AI assistance. The author is responsible for verifying the accuracy of the content and for the final manuscript.

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